JOURNAL OF APPROXIMATION THEORY 50, 341–358 (1987)

Approximation by Nörlund Means of Double Fourier Series for Lipschitz Functions*

F. MÓRICZ

University of Szeged, Bolyai Institute, Aradi Vértanúk tere 1, 6720 Szeged, Hungary

AND

B. E. RHOADES

Indiana University, Department of Mathematics, Bloomington, Indiana 47405, U.S.A.

Communicated by R. Bojanic

Received July 5, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

We study the rate of uniform approximation by Nörlund means of the rectangular partial sums of the double Fourier series of a function $f(x, y)$ belonging to the class Lipα, $0 < \alpha \leqslant 1$, on the two-dimensional torus $-\pi < x, y \leqslant \pi$. As a special case we obtain the rate of uniform approximation by double Cesàro means.

1. NÖRLUND SUMMABILITY OF DOUBLE NUMERICAL SEQUENCES

Let $\mathscr{P} = \{p_{jk}: j, k = 0, 1,...\}$ be a double sequence of nonnegative numbers, $p_{00} > 0$. Set

$$P_{mn} = \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} \qquad (m, n = 0, 1,...).$$

Given a double sequence $\{s_{jk}: j, k = 0, 1,...\}$ of complex numbers, the Nörlund means t_{mn} are defined by

$$t_{mn} = \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} p_{m-j,n-k} s_{jk} \qquad (m, n = 0, 1,...).$$

* This research was completed while the first author was a visiting professor at Indiana University, Bloomington, Indiana, U.S.A., during the academic year 1983–1984.

0021-9045/87 $3.00

We say that the Nörlund method generated by $\mathscr{P}$, or simply the $\mathscr{P}$-method of summability, is regular if whenever s_{mn} tends to a finite limit s as $m, n \to \infty$ and the s_{mn} are bounded for $m, n = 0, 1, \ldots$, then t_{mn} also tends to the same limit s as $m, n \to \infty$.

THEOREM A [3, p. 39]. *If* $\mathscr{P} = \{p_{jk} \geqslant 0: j, k = 0, 1, \ldots; p_{00} > 0\}$, *then the necessary and sufficient conditions for the regularity of the* $\mathscr{P}$-*method of summability are*

$$\lim_{m,n \to \infty} \frac{1}{P_{mn}} \sum_{k=0}^{n} p_{m-j,k} = 0 \qquad (j = 0, 1, \ldots; m \geqslant j)$$

and

$$\lim_{m,n \to \infty} \frac{1}{P_{mn}} \sum_{j=0}^{m} p_{j,n-k} = 0 \qquad (k = 0, 1, \ldots; n \geqslant k).$$

The (C, β, γ)-summability, $\beta, \gamma > -1$, is a particular case of the Nörlund summability, where $\mathscr{P} = \{p_{jk}\}$ is given by

$$p_{jk} = A_j^{\beta-1} A_k^{\gamma-1} \qquad (j, k = 0, 1, \ldots)$$

(even this is a factorable case), where

$$A_l^\alpha = \binom{\alpha + l}{l} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+l)}{l!}$$

for $l = 1, 2, \ldots$ and $A_0^\alpha = 1$. Then, as is known,

$$P_{mn} = A_m^\beta A_n^\gamma \qquad (m, n = 0, 1, \ldots).$$

Furthermore, there exist two positive constants C_1 and C_2 such that

$$C_1 \leqslant \frac{A_l^\alpha}{(l+1)^\alpha} \leqslant C_2 \qquad (l = 0, 1, \ldots; \alpha > -1)$$

(see, e.g., [5, p. 77]).

2. Nörlund Means for Double Fourier Series

Let $f(x, y)$ be a complex-valued function defined on the two-dimensional real torus Q: $-\pi < x \leqslant \pi$, $-\pi < y \leqslant \pi$. If $f \in L^1(Q)$, then its double Fourier series is

$$f(x, y) \sim \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)} \tag{2.1}$$

where

$$c_{jk}=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(s,t)\,e^{-i(js+kt)}ds\,dt \qquad (j,k=...,-1,0,1,...).$$

We associate with (2.1) the double sequence of (symmetric) rectangular partial sums

$$s_{mn}(x,y)=\sum_{j=-m}^{m}\sum_{k=-n}^{n} c_{jk}\,e^{i(jx+ky)} \qquad (m,n=0,1,...).$$

Now, the Nörlund means for (2.1) are defined as those for the sequence $\{s_{mn}(x,y)\}$:

$$t_{mn}(x,y)=\frac{1}{P_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n} p_{m-j,n-k}\,s_{jk}(x,y) \qquad (m,n=0,1,...).$$

The representation

$$t_{mn}(x,y)=\frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+s,y+t)\,K_{mn}(s,t)\,ds\,dt \tag{2.2}$$

plays a central role, where the Nörlund kernel $K_{mn}(x,t)$ is defined by

$$K_{mn}(s,t)=\frac{1}{P_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n} p_{m-j,n-k}\,D_j(s)\,D_k(t) \qquad (m,n=0,1,...), \tag{2.3}$$

and $D_j(s)$ and $D_k(t)$ are the Dirichlet kernels in terms of s and t, respectively, e.g.,

$$D_j(s)=\frac{1}{2}+\sum_{\nu=1}^{j}\cos \nu s=\frac{\sin(j+\frac{1}{2})s}{2\sin\frac{1}{2}s} \qquad (j=0,1,...).$$

From (2.2) it follows immediately that

$$t_{mn}(x,y)-f(x,y)=\frac{4}{\pi^2}\int_{0}^{\pi}\int_{0}^{\pi} \phi_{xy}(s,t)\,K_{mn}(s,t)\,ds\,dt \tag{2.4}$$

where

$$\phi_{xy}(s,t)=\tfrac{1}{4}\{f(x+s,y+t)+f(x-s,y+t)+f(x+s,y-t)+f(x-s,y-t)-4f(x,y)\}.$$

We say that the function f satisfies a Lipschitz condition of order $\alpha > 0$, in symbols $f \in \operatorname{Lip} \alpha$, if

$$\omega(\delta; f) = \sup_{(x, y) \in Q} \sup_{\{s^2 + t^2\}^{1/2} \leqslant \delta} |f(x+s, y+t) - f(x, y)| \leqslant C\delta^{\alpha} \qquad (\delta > 0) \tag{2.5}$$

with a constant C independent of δ. The quantity $\omega(\delta; f)$ is called the (total) modulus of continuity of f. As usual, we consider f as defined over the two-dimensional real Euclidean space $\mathbb{R}^2$ extended periodically in each variable (with period 2π).

Clearly, if $f \in \operatorname{Lip} \alpha$ for some $\alpha > 0$, then f is necessarily continuous everywhere. Only the case $0 < \alpha \leqslant 1$ is interesting. If $\alpha > 1$, then $\partial f / \partial x$ and $\partial f / \partial y$ exist and are zero everywhere, so f must be a constant.

Condition (2.5) can be rewritten as

$$|f(x+s, y+t) - f(x, y)| \leqslant C\{s^2 + t^2\}^{\alpha/2}$$

for every real x, y, s, and t; or equivalently,

$$|f(x+s, y+t) - f(x, y)| \leqslant C(|s|^{\alpha} + |t|^{\alpha}). \tag{2.6}$$

Indeed, for every real s, t and $0 < \alpha \leqslant 2$

$$\{s^2 + t^2\}^{\alpha/2} \leqslant |s|^{\alpha} + |t|^{\alpha} \leqslant 2\{s^2 + t^2\}^{\alpha/2}.$$

Here the first inequality is the Minkowski one, while the second is trivial.

Condition (2.6) obviously yields

$$|\phi_{xy}(s, t)| \leqslant C(|s|^{\alpha} + |t|^{\alpha}). \tag{2.7}$$

During the proofs we actually use inequality (2.7) which is, in certain cases, weaker than (2.6).

3. Main Results

We will use the notations

$$\Delta_{10} p_{jk} = p_{jk} - p_{j+1,k},$$

$$\Delta_{01} p_{jk} = p_{jk} - p_{j,k+1},$$

and

$$\Delta_{11} p_{jk} = p_{jk} - p_{j+1,k} - p_{j,k+1} + p_{j+1,k+1} \qquad (j, k = 0, 1, \ldots).$$

The double sequence $\{p_{jk}\}$ is nondecreasing if $\Delta_{10} p_{jk} \leqslant 0$ and $\Delta_{01} p_{jk} \leqslant 0$, and is nonincreasing if $\Delta_{10} p_{jk} \geqslant 0$ and $\Delta_{01} p_{jk} \geqslant 0$ for every $j, k = 0, 1, \ldots$. We also set

$$q_{mn} = \frac{1}{P_{mn}} \sum_{k=0}^{n} p_{mk},$$

$$r_{mn} = \frac{1}{P_{mn}} \sum_{j=0}^{m} p_{jn} \qquad (m, n = 0, 1, \ldots).$$

First we consider the case where p_{jk} is nondecreasing. Then

$$(m+1)\, q_{mn} = \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} p_{mk}$$

$$\geqslant \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} = 1, \tag{3.1}$$

and similarly,

$$(n+1)\, r_{mn} \geqslant 1. \tag{3.2}$$

We also have

$$P_{mn} \leqslant (m+1)(n+1)\, p_{mn} \qquad (m, n = 0, 1, \ldots).$$

In the sequel, we need the opposite inequality:

$$\frac{(m+1)(n+1)\, p_{mn}}{P_{mn}} = O(1). \tag{3.3}$$

This condition is satisfied, for example, if p_{jk} has a power growth both in j and in k; i.e.,

$$p_{jk} = (j+1)^{\beta}(k+1)^{\gamma} \qquad \text{for some } \beta, \gamma \geqq 0.$$

Now, condition (3.3) implies that

$$(m+1)\, q_{mn} = \frac{m+1}{P_{mn}} \sum_{k=0}^{n} p_{mk}$$

$$\leqslant \frac{m+1}{P_{mn}} (n+1)\, p_{mn} = O(1) \tag{3.4}$$

and

$$(n+1)\, r_{mn} = O(1). \tag{3.5}$$

In particular, the conditions of regularity are satisfied:

$$\lim_{m,n\to\infty} q_{mn} = \lim_{m,n\to\infty} r_{mn} = 0.$$

Thus, we may assume that

$$q_{mn} < \pi \quad \text{and} \quad r_{mn} < \pi \quad (m, n = 0, 1, \ldots).$$

THEOREM 1. *Let* $\{p_{jk} > 0: j, k = 0, 1, \ldots\}$ *be a nondecreasing double sequence such that* $\Delta_{11} p_{jk}$ *is of fixed sign and condition* (3.3) *is satisfied. If* $f \in \operatorname{Lip}\alpha$, $0 < \alpha < 1$, *then*

$$\sup_{(x,y)\in Q} |t_{mn}(x, y) - f(x, y)| = O(q_{mn}^{\alpha} + r_{mn}^{\alpha}) \qquad \text{if} \quad 0 < \alpha < 1,$$

$$= O\left(q_{mn} \log \frac{\pi}{q_{mn}} + r_{mn} \log \frac{\pi}{r_{mn}}\right) \qquad \text{if} \quad \alpha = 1 \tag{3.6}$$

Second we treat the case where p_{jk} is nonincreasing. Then

$$(m+1)\, q_{mn} \leqslant 1 \quad \text{and} \quad (n+1)\, r_{mn} \leqslant 1 \tag{3.7}$$

(cf. (3.1) and (3.2)).

THEOREM 2. *Let* $\{p_{jk} \geqslant 0: j, k = 0, 1, \ldots; \ p_{00} > 0\}$ *be a nonincreasing double sequence such that* $\Delta_{11} p_{jk}$ *is of fixed sign. If* $f \in \operatorname{Lip}\alpha$, $0 < \alpha \leqslant 1$, *then*

$$\sup_{(x,y)\in Q} |t_{mn}(x, y) - f(x, y)|$$

$$= O\left\{\frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} \left(\frac{P_{jk}}{(j+1)^{\alpha+1}(k+1)} + \frac{P_{jk}}{(j+1)(k+1)^{\alpha+1}}\right)\right\}. \tag{3.8}$$

In the special case where

$$\lim_{m,n\to\infty} p_{mn} > 0, \tag{3.9}$$

we have

$$\frac{1}{(m+1)\, q_{mn}} \leqslant \frac{p_{00}}{p_{mm}} = O(1) \quad \text{and} \quad \frac{1}{(n+1)\, r_{mn}} = O(1)$$

and the right-hand side of (3.8) reduces to that of (3.6).

COROLLARY 1. *Let* $\{p_{jk}>0\colon j, k=0, 1, \ldots\}$ *be a nonincreasing double sequence such that* $\Delta_{11} p_{jk}$ *is of fixed sign and condition* (3.9) *is satisfied. If* $f \in \operatorname{Lip} \alpha$, $0<\alpha \leqslant 1$, *then statement* (3.6) *holds.*

The approximation rate for (C, β, γ)-summability immediately follows from Theorem 1 (for $\beta, \gamma \geqslant 1$) and Theorem 2 (for $\alpha \leqslant \beta$, $\gamma \leqslant 1$).

COROLLARY 2. *If* $f \in \operatorname{Lip} \alpha$, $0<\alpha \leqslant 1$, *and* $\beta, \gamma \geqslant \alpha$, *then*

$$\sup_{(x, y) \in Q}\left|\frac{1}{A_m^\beta A_n^\gamma} \sum_{j=0}^m \sum_{k=0}^n A_{m-j}^{\beta-1} A_{n-k}^{\gamma-1} s_{jk}(x, y)-f(x, y)\right|$$

$$=O\left(\frac{1}{(m+1)^\alpha}+\frac{1}{(n+1)^\alpha}\right) \qquad \text{if} \quad \beta>\alpha \text{ and } \gamma>\alpha,$$

$$=O\left(\frac{\log(m+2)}{(m+1)^\alpha}+\frac{1}{(n+1)^\alpha}\right) \qquad \text{if} \quad \beta=\alpha \text{ and } \gamma>\alpha,$$

$$=O\left(\frac{\log(m+2)}{(m+1)^\alpha}+\frac{\log(n+2)}{(n+1)^\alpha}\right) \qquad \text{if} \quad \beta=\gamma=\alpha.$$

Theorem 1 is an extension of that announced by T. Singh (see [2, p. 364]) from the one-dimensional case to the two-dimensional case, while Theorem 2 is an extension of that in [1]. Our method clearly applies to higher-dimensional Fourier series as well. The extensions of our results to d-dimensional cases, where d is an integer greater than 2, are straightforward.

4. ESTIMATION OF THE NÖRLUND KERNEL

We will use some well-known estimates. For $j=0, 1, \ldots$

$$|D_j(s)|<j+1 \qquad \text{for every } s. \tag{4.1}$$

For $a, b=0, 1, \ldots;\ a \leqslant b$,

$$\sum_{j=a}^b \sin\left(j+\frac{1}{2}\right) s=\frac{\cos as-\cos(b+1) s}{2 \sin \frac{1}{2} s},$$

whence, on account of the inequality

$$\frac{\sin s}{s} \geqslant \frac{2}{\pi} \qquad \text{for} \quad 0<s \leqslant \frac{\pi}{2}, \tag{4.2}$$

we obtain

$$\left|\sum_{j=a}^{b} \sin\left(j+\frac{1}{2}\right)s\right| \leqslant \frac{\pi}{s} \qquad \text{for} \quad 0<s\leqslant \pi. \tag{4.3}$$

Similarly,

$$\left|\sum_{j=a}^{b} D_j(s)\right| = \left|\frac{\cos as - \cos(b+1)s}{(2\sin\frac{1}{2}s)^2}\right|$$
$$\leqslant \frac{\pi^2}{2s^2} \qquad \text{for} \quad 0<s\leqslant \pi. \tag{4.4}$$

We note that $1/(b+1)\sum_{j=0}^{b} D_j(s)$ is the Fejér kernel (cf. [5, pp. 49, 88]). The Nörlund kernel $K_{mn}(s, t)$ is defined by (2.3).

LEMMA 1. *Let $\{p_{jk}>0; j, k=0, 1,...\}$ be a nondecreasing double sequence such that $\Delta_{11} p_{jk}$ is of fixed sign. Then*

$$\begin{aligned}
|K_{mn}(s, t)| &\leqslant (m+1)(n+1) && \textit{for every } s \textit{ and } t,\\
&\leqslant \frac{\pi^2}{2}\frac{1}{P_{mn}s^2}\sum_{k=0}^{n}(k+1)\,p_{m,n-k} && \textit{for every } t \textit{ and } 0<s\leqslant\pi,\\
&\leqslant \frac{\pi^2}{2}\frac{1}{P_{mn}t^2}\sum_{j=0}^{m}(j+1)\,p_{m-j,n} && \textit{for every } s \textit{ and } 0<t\leqslant\pi,\\
&\leqslant \frac{3\pi^4}{4}\frac{p_{mn}}{P_{mn}s^2t^2} && \textit{for every } 0<s,\, t\leqslant\pi. \qquad (4.5)
\end{aligned}$$

Proof. By (4.1),

$$\begin{aligned}
|K_{mn}(s, t)| &\leqslant \frac{1}{P_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n} p_{m-j,n-k}|D_j(s)|\;|D_k(t)|\\
&\leqslant \frac{1}{P_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n}(j+1)(k+1)\,p_{m-j,n-k}\\
&= \frac{1}{P_{mn}}\sum_{j=0}^{m}\sum_{k=0}^{n} P_{jk}\leqslant (m+1)(n+1),
\end{aligned}$$

which is (4.5i). The monotonicity of the p_{jk} is not used here.

Again from (4.1),

$$\begin{aligned}
P_{mn}|K_{mn}(s, t)| &\leqslant \sum_{k=0}^{n}\left|\sum_{j=0}^{m} p_{m-j,n-k}D_j(s)\right|\;|D_k(t)|\\
&\leqslant \sum_{k=0}^{n}(k+1)\left|\sum_{j=0}^{m} p_{m-j,n-k}D_j(s)\right|. \qquad (4.6)
\end{aligned}$$

For each k, we rewrite the inner sum by an Abel's transformation (see, e.g., [5, p. 3]) as

$$\sum_{j=0}^{m} p_{m-j,n-k} D_j(s) = -\sum_{j=1}^{n} \Delta_{10} p_{m-j,n-k} \sum_{l=0}^{j-1} D_l(s) + p_{0,n-k} \sum_{l=0}^{m} D_l(s),$$

whence, by (4.4) and the assumption that p_{jk} is nondecreasing in j, we get

$$\left|\sum_{j=0}^{m} p_{m-j,n-k} D_j(s)\right| \leqslant \frac{\pi^2}{2s^2}\left(\sum_{j=1}^{m} \Delta_{10} p_{m-j,n-k} + p_{0,n-k}\right) = \frac{\pi^2}{2s^2} p_{m,n-k}. \tag{4.7}$$

Combining (4.6) and (4.7) yields (5.4ii).

Equation (4.5iii) can be shown in a similar way.

To prove (4.5iv), we first perform a double Abel's transformation (see, e.g., [4]):

$$\begin{aligned} P_{mn} K_{mn}(s, t) &= \sum_{j=1}^{m} \sum_{k=1}^{n} \Delta_{11} p_{m-j,n-k} \sum_{a=0}^{j-1} D_a(s) \sum_{b=0}^{k-1} D_b(t) \\ &\quad - \sum_{j=1}^{m} \Delta_{10} p_{m-j,0} \sum_{a=0}^{j-1} D_a(s) \sum_{b=0}^{n} D_b(t) \\ &\quad - \sum_{k=1}^{n} \Delta_{01} p_{0,n-k} \sum_{a=0}^{m} D_a(s) \sum_{b=0}^{k-1} D_b(t) \\ &\quad + p_{00} \sum_{a=0}^{m} D_a(s) \sum_{b=0}^{n} D_b(t), \end{aligned} \tag{4.8}$$

whence, by (4.4),

$$\begin{aligned} P_{mn} |K_{mn}(s, t)| &\leqslant \frac{\pi^4}{4s^2t^2}\left(\sum_{j=1}^{m} \sum_{k=1}^{n} |\Delta_{11} p_{m-j,n-k}| \right. \\ &\quad \left. + \sum_{j=1}^{m} \Delta_{10} p_{m-j,0} + \sum_{k=1}^{n} \Delta_{01} p_{0,n-k} + p_{00}\right). \end{aligned} \tag{4.9}$$

Since $\Delta_{11} p_{jk}$ is of fixed sign,

$$\sum_{j=1}^{m} \sum_{k=1}^{n} |\Delta_{11} p_{m-j,n-k}| = \left| \sum_{j=1}^{m} \sum_{k=1}^{n} \Delta_{11} p_{m-j,n-k} \right|$$
$$= |p_{mn} - p_{m0} - p_{0n} - p_{00}|.$$

Returning to (4.9), if $\Delta_{11} p_{jk} \geqslant 0$,

$$P_{mn} |K_{mn}(s, t)| \leqslant \frac{\pi^4}{4s^2t^2} [(p_{mn} - p_{m0} - p_{0n} + p_{00})$$
$$+ (p_{m0} - p_{00}) + (p_{0n} - p_{00}) + p_{00}]$$
$$= \frac{\pi^4 p_{mn}}{4s^2t^2},$$

while if $\Delta_{11} p_{jk} \leqslant 0$,

$$P_{mn} |K_{mn}(s, t)| \leqslant \frac{\pi^4}{4s^2t^2} [(-p_{mn} + p_{m0} + p_{0n} - p_{00})$$
$$+ (p_{m0} - p_{00}) + (p_{0n} - p_{00}) + p_{00}]$$
$$= \frac{\pi^4}{4s^2t^2} (-p_{mn} + 2p_{m0} + 2p_{0n} - 2p_{00})$$
$$\leqslant \frac{3\pi^4}{4s^2t^2} p_{mn}.$$

LEMMA 2. *Let* $\{p_{jk} \geqslant 0\colon j, k = 0, 1,...;\ p_{00} > 0\}$ *be a nonincreasing double sequence such that* $\Delta_{11} p_{jk}$ *is of fixed sign, and let* $\sigma = [1/s]$, $\tau = [1/t]$ *where* $[\cdot]$ *means the integral part. Then*

$$|K_{mn}(s, t)| \leqslant (m+1)(n+1) \qquad \textit{for every} \quad s \textit{ and } t,$$
$$\leqslant \frac{\pi(\pi+1)}{2} \frac{1}{P_{mn} s} \sum_{k=0}^{n} (k+1) \sum_{j=0}^{\sigma} p_{j,n-k} \qquad \textit{for every} \quad t \textit{ and } 0 < s \leqslant \pi,$$
$$\leqslant \frac{\pi(\pi+1)}{2} \frac{1}{P_{mn} t} \sum_{j=0}^{m} (j+1) \sum_{k=0}^{\tau} p_{m-j,k} \qquad \textit{for every} \quad s \textit{ and } 0 < t \leqslant \pi,$$
$$\leqslant \frac{\pi^2(1 + 2\pi + 3\pi^2)}{4} \frac{P_{\sigma\tau}}{P_{mn} st} \qquad \textit{for every} \quad 0 < s,\ t \leqslant \pi. \tag{4.10}$$

Proof. Equation (4.10i) coincides with (4.5i), which holds without any monotonicity condition, as we remarked in the proof of Lemma 1.

By (4.6) and (4.2),

$$P_{mn}|K_{mn}(s,t)| \leqslant \sum_{k=0}^{n} (k+1) \left| \sum_{j=0}^{m} p_{m-j,n-k} D_j(s) \right|$$
$$\leqslant \frac{\pi}{2s} \sum_{k=0}^{n} (k+1) \left| \sum_{j=0}^{m} p_{j,n-k} \sin\left(m-j+\frac{1}{2}\right)s \right|. \tag{4.11}$$

A simple estimate shows that, for each k,

$$\left| \sum_{j=0}^{m} p_{j,n-k} \sin\left(m-j+\frac{1}{2}\right)s \right|$$
$$\leqslant \sum_{j=0}^{\sigma} p_{j,n-k} + \left| \sum_{j=\sigma+1}^{m} p_{j,n-k} \sin\left(m-j+\frac{1}{2}\right)s \right|. \tag{4.12}$$

Using an Abel's transformation,

$$\sum_{j=\sigma+1}^{m} p_{j,n-k} \sin\left(m-j+\frac{1}{2}\right)s$$
$$= \sum_{j=\sigma+1}^{m-1} \Delta_{10} p_{j,n-k} \sum_{l=\sigma+1}^{j} \sin\left(m-l+\frac{1}{2}\right)s$$
$$+ p_{m,n-k} \sum_{l=\sigma+1}^{m} \sin\left(m-l+\frac{1}{2}\right)s. \tag{4.13}$$

From (4.3), the fact that p_{jk} is nonincreasing in j, and that $1/s < \sigma + 1$, we can conclude that

$$\left| \sum_{j=\sigma+1}^{m} p_{j,n-k} \sin\left(m-j+\frac{1}{2}\right)s \right| \leqslant \frac{\pi}{s} p_{\sigma+1,n-k}$$
$$\leqslant \pi(\sigma+1)\, p_{\sigma+1,n-k} \leqslant \pi \sum_{j=0}^{\sigma} p_{j,n-k}. \tag{4.14}$$

Now, the combination of (4.11), (4.12), and (4.14) provides (4.10ii).

Equation (4.10iii) can be deduced similarly.

To prove (4.10iv), by (4.2) we begin with the inequality

$$P_{mn}|K_{mn}(s,t)|$$
$$= \left| \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} D_{m-j}(s)\, D_{n-k}(t) \right|$$
$$\leqslant \frac{\pi^2}{4st} \left| \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} \sin\left(m-j+\frac{1}{2}\right)s \sin\left(n-k+\frac{1}{2}\right)t \right|. \tag{4.15}$$

We divide the double sum into four parts:

$$\left| \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} \sin\left(m-j+\frac{1}{2}\right) s \sin\left(n-k+\frac{1}{2}\right) t \right|$$

$$\leqslant \sum_{j=0}^{\sigma} \sum_{k=0}^{\tau} p_{jk} + \sum_{k=0}^{\tau} \left| \sum_{j=\sigma+1}^{m} p_{jk} \sin\left(m-j+\frac{1}{2}\right) s \right|$$

$$+ \sum_{j=0}^{\sigma} \left| \sum_{k=\tau+1}^{n} p_{jk} \sin\left(n-k+\frac{1}{2}\right) t \right|$$

$$+ \left| \sum_{j=\sigma+1}^{m} \sum_{k=\tau+1}^{n} p_{jk} \sin\left(m-j+\frac{1}{2}\right) s \sin\left(n-k+\frac{1}{2}\right) t \right|$$

$$= P_{\sigma\tau} + A_1 + A_2 + A_3, \qquad \text{say.} \tag{4.16}$$

For A_1, we can perform an Abel's transformation similar to (4.13) and conclude that

$$\left| \sum_{j=\sigma+1}^{m} p_{jk} \sin\left(m-j+\frac{1}{2}\right) s \right|$$

$$\leqslant \sum_{j=\sigma+1}^{m-1} \Delta_{10} p_{jk} \left| \sum_{l=\sigma+1}^{j} \sin\left(m-l+\frac{1}{2}\right) s \right|$$

$$+ p_{mk} \left| \sum_{l=\sigma+1}^{m} \sin\left(m-l+\frac{1}{2}\right) s \right|$$

$$\leqslant \frac{\pi}{s} p_{\sigma+1,k} \leqslant \pi(\sigma+1)\, p_{\sigma+1,k} \leqslant \pi \sum_{j=0}^{\sigma} p_{jk}$$

(cf. (4.14)), which results in

$$A_1 \leqslant \pi P_{\sigma\tau}. \tag{4.17}$$

Analogously,

$$A_2 \leqslant \pi P_{\sigma\tau}. \tag{4.18}$$

For A_3, we perform a double Abel's transformation (cf. (4.8)):

$$\sum_{j=\sigma+1}^{m} \sum_{k=\tau+1}^{n} p_{jk} \sin\left(m-j+\frac{1}{2}\right) s \sin\left(n-k+\frac{1}{2}\right) t$$

$$= \sum_{j=\sigma+1}^{m-1} \sum_{k=\tau+1}^{n-1} \Delta_{11} p_{jk} \sum_{a=\sigma+1}^{j} \sin\left(m-a+\frac{1}{2}\right) s \sum_{b=\tau+1}^{k} \sin\left(n-b+\frac{1}{2}\right) t$$

$$+\sum_{j=\sigma+1}^{m-1} \Delta_{10} p_{jn} \sum_{a=\sigma+1}^{j} \sin\left(m-a+\frac{1}{2}\right)s \sum_{b=\tau+1}^{n} \sin\left(n-b+\frac{1}{2}\right)t$$

$$+\sum_{k=\tau+1}^{n-1} \Delta_{01} p_{mk} \sum_{a=\sigma+1}^{m} \sin\left(m-a+\frac{1}{2}\right)s \sum_{b=\tau+1}^{k} \sin\left(n-b+\frac{1}{2}\right)t$$

$$+p_{mn} \sum_{a=\sigma+1}^{m} \sin\left(m-a+\frac{1}{2}\right)s \sum_{b=\tau+1}^{n} \sin\left(n-b+\frac{1}{2}\right)t,$$

whence, by (4.3),

$$\left| \sum_{j=\sigma+1}^{m} \sum_{k=\tau+1}^{n} p_{jk} \sin\left(m-j+\frac{1}{2}\right)s \sin\left(n-k+\frac{1}{2}\right)t \right|$$

$$\leqslant \frac{\pi^2}{st}\left\{\left| \sum_{j=\sigma+1}^{m} \sum_{k=\tau+1}^{n} \Delta_{11} p_{jk}\right| + \sum_{j=\sigma+1}^{m-1} \Delta_{10} p_{jn} + \sum_{k=\tau+1}^{n-1} \Delta_{01} p_{mk} + p_{mn}\right\}$$

$$= \frac{\pi^2}{st} p_{\sigma+1,\tau+1} \qquad \text{if} \quad \Delta_{11} p_{jk} \geqslant 0,$$

$$= \frac{\pi^2}{st}\left(-2p_{mn} + 2p_{\sigma+1,n} + 2p_{m,\tau+1} - p_{\sigma+1,\tau+1}\right)$$

$$\leqslant \frac{3\pi^2}{st} p_{\sigma+1,\tau+1} \qquad \text{if} \quad \Delta_{11} p_{jk} \leqslant 0.$$

Thus, in any case,

$$A_3 \leqslant \frac{3\pi^2}{st} p_{\sigma+1,\tau+1} \leqslant 3\pi^2(\sigma+1)(\tau+1)\, p_{\sigma+1,\tau+1}$$

$$\leqslant 3\pi^2 \sum_{j=0}^{\sigma} \sum_{k=0}^{\tau} p_{jk} = 3\pi^2 P_{\sigma\tau}. \tag{4.19}$$

Putting (4.16)–(4.19) together yields

$$\left| \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk} \sin\left(m-j+\frac{1}{2}\right)s \sin\left(n-k+\frac{1}{2}\right)t \right|$$

$$\leqslant (1+2\pi+3\pi^2)\, P_{\sigma\tau}.$$

Hence (4.15) immediately implies (4.10iv).

5. Proofs of the Theorems

Proof of Theorem 1. We start with representation (2.4), decomposing the integral as follows:

$$\frac{\pi^2}{4}\,|t_{mn}(x, y)-f(x, y)|$$

$$\leqslant \left\{ \int_0^{q_{mn}}\int_0^{r_{mn}} + \int_{q_{mn}}^{\pi}\int_0^{r_{mn}} + \int_0^{q_{mn}}\int_{r_{mn}}^{\pi} + \int_{q_{mn}}^{\pi}\int_{r_{mn}}^{\pi} \right\} |\phi_{xy}(s, t)|\,|K_{mn}(s, t)|\,ds\,dt$$

$$= I_1 + I_2 + I_3 + I_4, \qquad \text{say.} \tag{5.1}$$

Each time $\phi_{xy}(s, t)$ is estimated by (2.7) and the appropriate estimate of Lemma 1 is substituted forthe kernel $K_{mn}(s, t)$.

By (4.5), for $\alpha > 0$

$$I_1 \leqslant (m+1)(n+1)\int_0^{q_{mn}}\int_0^{r_{mn}} (s^\alpha + t^\alpha)\,ds\,dt$$

$$= \frac{1}{\alpha+1}\,(m+1)(n+1)\,q_{mn}r_{mn}(q_{mn}^\alpha + r_{mn}^\alpha).$$

By (3.4) and (3.5),

$$I_1 = O(q_{mn}^\alpha + r_{mn}^\alpha). \tag{5.2}$$

By (4.5ii),

$$I_2 \leqslant \frac{\pi^2}{2}\frac{1}{P_{mn}}\sum_{k=0}^{n}(k+1)\,p_{m,n-k}\int_{q_{mn}}^{\pi}\int_0^{r_{mn}}\frac{s^\alpha + t^\alpha}{s^2}\,dt\,ds,$$

whence for $0 < \alpha < 1$,

$$I_2 \leqslant \frac{\pi^2}{2}\frac{r_{mn}}{q_{mn}P_{mn}}\sum_{k=0}^{n}(k+1)\,p_{m,n-k}\left(\frac{q_{mn}^\alpha}{1-\alpha} + \frac{r_{mn}^\alpha}{\alpha+1}\right),$$

while for $\alpha = 1$,

$$I_2 \leqslant \frac{\pi^2}{2}\frac{r_{mn}}{q_{mn}P_{mn}}\sum_{k=0}^{n}(k+1)\,p_{m,n-k}\left(q_{mn}\log\frac{\pi}{q_{mn}} + \frac{1}{2}r_{mn}\right).$$

Using (3.5),

$$\frac{r_{mn}}{q_{mn}P_{mn}} \sum_{k=0}^{n} (k+1)\, p_{m,n-k} \leqslant \frac{r_{mn}}{q_{mn}P_{mn}} (n+1) \sum_{k=0}^{n} p_{mk}$$
$$= (n+1)\, r_{mn} = O(1).$$

So,

$$\begin{aligned} I_2 &= O(q_{mn}^{\alpha} + r_{mn}^{\alpha}) && \text{if} \quad 0 < \alpha < 1, \\ &= O\left(q_{mn} \log \frac{\pi}{q_{mn}} + r_{mn}\right) && \text{if} \quad \alpha = 1. \end{aligned} \tag{5.3}$$

Similarly, this time using (4.5iii),

$$\begin{aligned} I_3 &= O(q_{mn}^{\alpha} + r_{mn}^{\alpha}) && \text{if} \quad 0 < \alpha < 1, \\ &= O\left(q_{mn} + r_{mn} \log \frac{\pi}{r_{mn}}\right) && \text{if} \quad \alpha = 1. \end{aligned} \tag{5.4}$$

By (4.5iv),

$$I_4 \leqslant \frac{3\pi^4}{4} \frac{p_{mn}}{P_{mn}} \int_{q_{mn}}^{\pi} \int_{r_{mn}}^{\pi} \frac{s^{\alpha} + t^{\alpha}}{s^2 t^2}\, ds\, dt,$$

whence for $0 < \alpha < 1$,

$$I_4 \leqslant \frac{3\pi^4}{4(1-\alpha)} \frac{p_{mn}}{q_{mn} r_{mn} P_{mn}} (q_{mn}^{\alpha} + r_{mn}^{\alpha}),$$

while for $\alpha = 1$,

$$I_4 \leqslant \frac{3\pi^4}{4} \frac{p_{mn}}{q_{mn} r_{mn} P_{mn}} \left(q_{mn} \log \frac{\pi}{q_{mn}} + r_{mn} \log \frac{\pi}{r_{mn}}\right).$$

By (3.1), (3.2), and (3.3),

$$\frac{p_{mn}}{q_{mn} r_{mn} P_{mn}} = \frac{(m+1)(n+1)\, p_{mn}}{(m+1)\, q_{mn}(n+1)\, r_{mn} P_{mn}} = O(1).$$

Consequently,

$$\begin{aligned} I_4 &= O(q_{mn}^{\alpha} + r_{mn}^{\alpha}) && \text{if} \quad 0 < \alpha < 1, \\ &= O\left(q_{mn} \log \frac{\pi}{q_{mn}} + r_{mn} \log \frac{\pi}{r_{mn}}\right) && \text{if} \quad \alpha = 1. \end{aligned} \tag{5.5}$$

Collecting (5.1)–(5.5) together yields (3.6).

Proof of Theorem 2. We use decomposition (5.1) with q_{mn} and r_{mn} replaced by $\pi/(m+1)$ and $\pi/(n+1)$, respectively. For brevity, we denote by Q_{mn} the quantity in braces on the right-hand side of (3.8).

By (4.10i), for $\alpha > 0$

$$I_1 \leqslant (m+1)(n+1) \int_0^{\pi/(m+1)} \int_0^{\pi/(n+1)} (s^\alpha + t^\alpha)\, ds\, dt$$

$$\leqslant \frac{\pi^{\alpha+2}}{\alpha+1} \left(\frac{1}{(m+1)^\alpha} + \frac{1}{(n+1)^\alpha} \right). \tag{5.6}$$

Since p_{jk} is nonincreasing, we trivially have

$$P_{jk} \geqslant (j+1)(k+1)\, p_{jk} \qquad (j, k = 0, 1, ...).$$

Therefore,

$$\frac{1}{(m+1)^\alpha} = \frac{1}{(m+1)^\alpha} \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} p_{jk}$$

$$\leqslant \frac{1}{(m+1)^\alpha} \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{P_{jk}}{(j+1)(k+1)}$$

$$\leqslant \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{P_{jk}}{(j+1)^{\alpha+1}(k+1)},$$

and similarly,

$$\frac{1}{(n+1)^\alpha} \leqslant \frac{1}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{P_{jk}}{(j+1)(k+1)^{\alpha+1}}.$$

Combining (5.6) with the last two inequalities results in

$$I_1 = O(Q_{mn}). \tag{5.7}$$

By (4.10ii),

$$I_2 \leqslant \frac{\pi(\pi+1)}{2P_{mn}} \sum_{k=0}^{n} (k+1)$$

$$\times \int_{\pi/(m+1)}^{\pi} \int_0^{\pi/(n+1)} \frac{s^\alpha + t^\alpha}{s} \sum_{j=0}^{\sigma} p_{j,n-k}\, dt\, ds$$

$$= \frac{\pi(\pi+1)}{2P_{mn}} \sum_{k=0}^{n} (k+1)$$

$$\times \left\{ \frac{\pi}{n+1} \int_{\pi/(m+1)}^{\pi} s^{\alpha-1} \sum_{j=0}^{\sigma} p_{j,n-k}\, ds \right.$$

$$\left. + \frac{\pi^{\alpha+1}}{(\alpha+1)(n+1)^{\alpha+1}} \int_{\pi/(m+1)}^{\pi} \frac{1}{s} \sum_{j=0}^{\sigma} p_{j,n-k}\, ds \right\}.$$

In each integration replace s by $1/u$ (remembering that $\sigma=[1/s]$) to get

$$I_2=\frac{O(1)}{P_{mn}}\sum_{k=0}^{n}(k+1)\left\{\frac{1}{n+1}\int_{1/\pi}^{(m+1)/\pi}\frac{1}{u^{\alpha+1}}\sum_{j=1}^{[u]}p_{j,n-k}\,du+\frac{1}{(n+1)^{\alpha+1}}\int_{1/\pi}^{(m+1)/\pi}\frac{1}{u}\sum_{j=1}^{[u]}p_{j,n-k}\,du\right\}.$$

Then making a simple approximation to the integrals involved yields

$$I_2=\frac{O(1)}{P_{mn}}\sum_{k=0}^{n}(k+1)\left\{\frac{1}{n+1}\sum_{l=0}^{m}\frac{1}{(l+1)^{\alpha+1}}\sum_{j=0}^{l}p_{j,n-k}+\frac{1}{(n+1)^{\alpha+1}}\sum_{l=0}^{m}\frac{1}{l+1}\sum_{j=0}^{l}p_{j,n-k}\right\}. \tag{5.8}$$

The first sum on the right is equal to

$$A=\frac{1}{(n+1)P_{mn}}\sum_{k=0}^{n}(k+1)\sum_{l=0}^{m}\frac{1}{(l+1)^{\alpha+1}}\sum_{j=0}^{l}p_{j,n-k}$$
$$=\frac{1}{(n+1)P_{mn}}\sum_{l=0}^{m}\frac{1}{(l+1)^{\alpha+1}}\sum_{j=0}^{l}\sum_{k=0}^{n}(k+1)\,p_{j,n-k}.$$

Using the identity

$$\sum_{k=0}^{n}(k+1)\,p_{j,n-k}=\sum_{k=0}^{n}\sum_{r=0}^{n-k}p_{jr},$$

we can write

$$A=\frac{1}{(n+1)P_{mn}}\sum_{l=0}^{m}\frac{1}{(l+1)^{\alpha+1}}\sum_{k=0}^{n}\sum_{j=0}^{l}\sum_{r=0}^{n-k}p_{jr}$$
$$=\frac{1}{(n+1)P_{mn}}\sum_{l=0}^{m}\sum_{k=0}^{n}\frac{P_{l,n-k}}{(l+1)^{\alpha+1}}$$
$$=\frac{1}{(n+1)P_{mn}}\sum_{l=0}^{m}\sum_{k=0}^{n}\frac{P_{lk}}{(l+1)^{\alpha+1}}$$
$$\leqslant\frac{1}{P_{mn}}\sum_{l=0}^{m}\sum_{k=0}^{n}\frac{P_{lk}}{(l+1)^{\alpha+1}(k+1)}. \tag{5.9}$$

The second sum in the right-hand side of (5.8) can be dominated in a similar manner:

$$\frac{1}{(n+1)^{\alpha+1}P_{mn}} \sum_{k=0}^{n} (k+1) \sum_{l=0}^{m} \frac{1}{l+1} \sum_{j=0}^{l} p_{j,n-k}$$

$$\leqslant \frac{1}{P_{mn}} \sum_{l=0}^{m} \sum_{k=0}^{n} \frac{P_{lk}}{(k+1)^{\alpha+1}(l+1)}. \tag{5.10}$$

From (5.8)–(5.10) it follows that

$$I_2 = O(Q_{mn}). \tag{5.11}$$

In an analogous way, by (4.10iii),

$$I_3 = O(Q_{mn}). \tag{5.12}$$

Using (4.10iv),

$$I_4 = \frac{O(1)}{P_{mn}} \int_{\pi/(m+1)}^{\pi} \int_{\pi/(n+1)}^{\pi} \frac{s^\alpha + t^\alpha}{st} P_{\sigma\tau}\, ds\, dt.$$

We replace s by $1/u$ and t by $1/v$, keeping in mind that $\sigma = [1/s]$ and $\tau = [1/t]$. As a result we obtain

$$I_4 = \frac{O(1)}{P_{mn}} \int_{1/\pi}^{(m+1)/\pi} \int_{1/\pi}^{(n+1)/\pi} \left(\frac{1}{u^{\alpha+1}v} + \frac{1}{uv^{\alpha+1}}\right) P_{[u],[v]}\, du\, dv.$$

A natural evaluation of this double integral shows that

$$I_4 = \frac{O(1)}{P_{mn}} \sum_{j=0}^{m} \sum_{k=0}^{n} \left(\frac{1}{(j+1)^{\alpha+1}(k+1)} + \frac{1}{(j+1)(k+1)^{\alpha+1}}\right) P_{jk} = O(Q_{mn}). \tag{5.13}$$

Combining (5.1), (5.7), (5.11)–(5.13) results in (3.8).

References

1. D. S. Goel and B. N. Sahney, On the degree of approximation of continuous functions, *Ranchi Univ. Math. J.* **4** (1973), 50–53.
2. A. B. S. Holland, A survey of degree of approximation of continuous functions, *SIAM Rev.* **23** (1981), 344–379.
3. C. N. Moore, Summable series and convergence factors, *Amer. Math. Soc. Colloq. Publ.* **22** (1938).
4. F. Móricz, The Kronecker lemmas for multiple series and some applications, *Acta Math. Acad. Sci. Hungar.* **37** (1981), 39–50.
5. A. Zygmund, "Trigonometric Series, I," Cambridge Univ. Press, Cambridge, 1959.